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A New Method of Determining the Primary Position of the Eye Using Listing's Law

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Abstract

A new method of determining the primary direction of regard, using Listing's law as a basis, is described. Previous techniques have required a trial-and-error experimental procedure. The present technique requires only a single measurement set of eye torsions in selected gaze directions. Experimental data to illustrate the technique are presented.

Key words: ocular primary position, measurement method, experimental results, Listing's law, eye torsion measurements, directions of gaze

What do we mean when we say the eyes are looking straight ahead? Search coils, contact lens levers, and so on can measure the fixation direction with remarkable precision. Still, we are left with a basic uncertainty in specifying a baseline direction, a primary direction.

A number of solutions to this problem have been proposed. They are cumbersome, however, or suffer from arbitrary factors. For example, a stereotaxic approach could be taken, identifying skull landmarks to specify an XYZ coordinate system. This technique, however, has proved too inaccurate for human neurosurgery,[1] and skull landmarks are hard to measure from without. Another approach is to note the position of the eye without innervation, under a neuromuscular block. Aside from medical and ethical difficulties, the eyes diverge in this state and the amount of the divergence varies with age.[2] Clearly this divergence does not tally with our common-sense view of "straight ahead."

These difficulties might lead one to question the intrinsic utility of the idea of a primary direction at all and to conclude that the primary direction is a matter of arbitrary definition. To understand muscle actions from a geometric and mechanical perspective, however, one needs a primary direction—if only to generate a coordinate system to quantify muscle positions, forces, attachments, and so on. All quantitative theories of muscle action[3-5] require such coordinates but leave unspecified the manner in which they are to be obtained. The present paper proposes a solution that is likely to have intrinsic merit, as well as wide applicability.

LISTING'S LAW

Under ordinary circumstances when the head is erect and the eye is engaged in steady fixation at distance, the globe has only 2 deg of rotational freedom. The 3rd

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deg of freedom is determined by Listing's law. This law states that the rotation state of the eye for any fixation is geometrically equivalent to the case in which the eye has rotated directly to the fixation position from the primary position, the fixation axis following a great-circle course. Stated in the most general terms, Listing's law asserts that there is a radial rotational symmetry in the behavior of the eye, such that all permissible rotational states can be considered equivalent to geodesic, or shortest-route, rotations from a primary axis of symmetry. By using the term *equivalent,* we emphasize that Listing's law makes no claim of describing how the eye actually arrived at its destination; it simply describes the final resting position, "as if" the eye took a great-circle course.

As a model for the illustration of Listing's law, consider a hemisphere glued to an evenly stretched membrane, which in turn is glued inside a rigid cylindrical frame (Fig. 1A). Attached to the hemisphere is a pointer that is at right angles to the membrane when the system is at rest.

One can rotate this hemisphere out of its resting position with a smooth rod, and in so doing will find that any rotational displacement will be in congruence with Listing's law (Fig. 1B). Instead of the 3 deg of freedom theoretically possessed by a freely rotating body, there are in this case only 2 deg of rotational freedom with 1 deg of rotational constraint. Each rotational displacement of the stick away from vertical will have associated with it a preferred value of rotation about the stick. Any other rotation about the pointer axis will not only bend the membrane but twist it as well. Because the twist requires an applied torque about the pointer axis and since one cannot apply this torque by a simple displacement of the pointer, the hemisphere and pointer will undergo no net twist as they go from their resting, or primary, position to any other.

So the immediate reason this particular model obeys Listing's law is a mechanical one; the 1 deg of rotational constraint is dictated by the radially symmetric attachments to the hemisphere. It should be noted that such a simple mechanical reason is not sufficient to explain Listing's law for the eye; the explanation requires neurological considerations as well.[6]

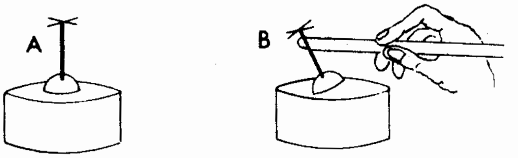

Fig. 1. Model of a body constructed so as to obey Listing's law. In the resting position (A), a hemisphere is glued to a rubber membrane, evenly stretched over a rigid circular frame. A pointer with a cross attached is attached to the hemisphere, set at right angles to the membrane when the system is at rest. By pushing the pointer with a rod (B), one can rotate the body with only 2 deg of freedom; one cannot rotate the hemisphere about the axis defined by the pointer. Such a mechanical configuration obeys Listing's law because any rotation away from the resting position (A) is not accompanied by a twist about the pointer axis.

Knowing Listing's law solves the problem of predicting eye torsion for any gaze direction if the primary direction is known. To find the primary direction is therefore the inverse problem.

Assuming, from Listing's law, that the line of sight coincides with the axis of rotational symmetry when the eye is in the primary position, it is, in principle, possible to determine the location of this primary direction from measurements of eye torsion and gaze direction.

Returning again to our ball-and-membrane model, the task of finding the primary direction is analogous to determining the direction that is perpendicular to the membrane when the ball is at rest—doing so, however, without any knowledge of the resting orientation of the membrane, relying only on a series of 3-dimensional measurements of the rotational state of the ball.

Geometric Reasoning

Before presenting the algebraic solution, I present an overall solution in geometric terms.

Consider a monocular spherical field of fixation, this being a reference sphere with an arbitrarily defined radius of unity. Any possible eye fixation can be considered as a point on this sphere; call this point the "visual pole." Furthermore, somewhere on this sphere and to be calculated by our method is the primary position, being determined by the intersection of the radial axis of symmetry and the reference sphere. This axis pierces the back of the reference sphere at a point called the "occipital pole." Now consider *any* circle on the reference sphere which passes through this occipital pole. Such circles have been defined by Helmholtz as direction circles, and they constitute the central concept for the ensuing analysis. Helmholtz showed that if Listing's law is correct, any fixation is analytically related to any other fixation by a unique circular trajectory that takes the visual pole along the direction circle joining these two fixation positions.[7-9] For any two fixation positions, there is one and only one simple circular trajectory (corresponding to a simple rotation) on the spherical field of fixation, taking the eye from one position to another in such a way that Listing's law is preserved. Furthermore, there is one and only one axis about which the eye can make such a simple rotation to bring the eye from one fixation point to the other, and this axis is normal to and passes through the center of this direction circle. Since this circle passes through the occipital point, the intersection of two of these circular trajectories between two pairs of fixation points will of necessity lie at the occipital point (Fig. 2). If we can calculate the locus of the unique circular trajectory linking two fixations, we can calculate the position of the occipital point, because it will lie at the intersection of two such circles. From this occipital point, one can easily determine the primary position, for it is the antipode of this occipital pole.

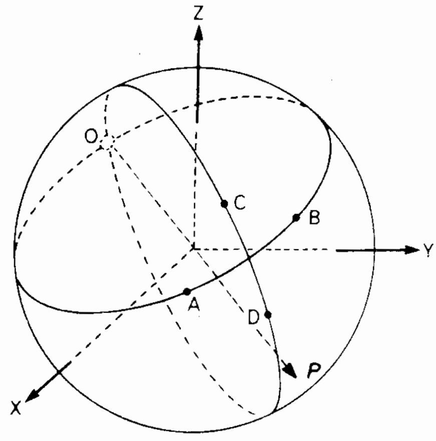

Fig. 2. Spherical field of fixation. The Y axis represents the reference direction. A, B, C, and D represent fixation points from which the calculation of the primary direction of regard is made. Such calculation requires the measurement of torsion for each gaze direction. Circles OCD and OAB represent the unique circular trajectories that connect pairs of gaze positions—AB and CD, respectively. If Listing's law is correct, these two circular trajectories are direction circles and the intersection of these circles is the occipital pole (O). Diametrically opposed to the occipital pole is the primary position of regard (P).

Algebraic Reasoning

To begin, choose a particular direction anywhere in front of the head. For convenience, the front of the apparatus will do. Have the eye look in this direction and note the torsion of a marker on the eye or retina when looking in this direction. Call this arbitrarily defined position of the eye, the reference position, α. Then measure the three dimensions of eye rotation for a number of gaze directions with respect to this reference position. Several techniques can be used, including photography,[10] afterimages,[11] search coils,[5] or contact lens mirror attachments.[12] The three parameters of the rotation can be expressed as Euler angles in a chosen coordinate system—a Fick system, for example—or, more conveniently, they can also be expressed as a 3x3 rotation matrix of direction cosines.[5, 10] For example, in Fig. 2, the Y axis represents the reference direction, and the rotational states of the eye corresponding to the gaze directions A, B, C, and D can be characterized by the matrices R_A, R_B, R_C, and R_D. Such matrices represent the transformations required to rotate the eye from the reference position, α, to any of the fixation positions measured.

Because rotation transformations form a group, it follows that compositions of such transformations or their inverses are themselves rotations and the multiplication of any two matrices corresponding to two rotations is itself a rotation; in particular, one rotation matrix correspnding to one fixation multiplied by the inverse of another specifies a rotation from the first fixation to the second. For example:

$$R_{AB} = R_A^{-1} R_B \qquad (1)$$

where R_{AB} is the rotation matrix linking the fixation change from A to B and where R_A^{-1} is the inverse of the matrix R_A. This matrix R_{AB} corresponds to the unique simple rotation transformation taking the eye from fixation A to fixation B.

This matrix enables one to ascertain the axis of the simple rotation that takes the eye from a given fixation to another. Such an axis vector $\bar{A}$ remains invariant under a rotation; thus,

$$R\bar{A} = \bar{A} \tag{2}$$

$$\begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \tag{3}$$

The solution to equation 2 is not unique but is in the form of $m(a_1a_2a_3)$, where a_1, a_2, a_3 represent direction cosines of a vector parallel to the axis of rotation and m is an arbitrary constant.[10] Now, given the XYZ coordinates of either of the fixation positions of a pair—point A, for example—one can obtain the equation of the plane that contains the direction circle joining the pair of fixations. This plane is defined as

$$a_1X + a_2Y + a_3Z + a_4 = 0 \tag{4}$$

where $a_4 = (-a_1x - a_2y - a_3z)$, and where $a_1a_2a_3$ represent components of the unit axis vector and x, y, z represent the XYZ coordinates of one of the pair of fixations.

We find the planes for two such pairs of gaze positions and the intersection of these two planes will form a line. The intersection of this line with the most posterior position of reference sphere establishes the occipital pole, which is equivalent to the intersection of two direction circles. Thus, the occipital pole can be described as a solution of the following three simultaneous equations:

$$a_1X + a_2Y + a_3Z + a_4 = 0 \tag{4}$$

$$a'_1X + a'_2Y + a'_3Z + a'_4 = 0 \tag{5}$$

$$X^2 + Y^2 + Z^2 = 1 \tag{6}$$

where equations 4 and 5 are the equations of the two planes and equation 6 is the equation of reference sphere. Since equation 6 is a quadratic, there are two solutions, corresponding to the two intersections of the two direction circles on the spherical field of fixation. The more posterior solution is the occipital pole; therefore, one must select the position where Y has the greatest negative value. This solution for the occipital point $\bar{O}(x, y, z)$ is the antipode of the primary position, $\bar{P}$:

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = -\bar{O} = \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix}$$

From these coordinates one can calculate the angular position of the primary position in Fick coordinates with

$$\theta = \tan^{-1}(p_1/p_2)$$

$$\phi = \tan^{-1}(p_3/\sqrt{p_1^2 + p_2^2})$$

In theory, the procedure outlined above is consistent and correct. In practice, however, one must consider noise, as measurements of gaze direction and torsion are not error-free. Consequently, the planes (equations 4 and 5) containing these direction circles could conceivably intersect outside the reference sphere, leading to imaginary roots, or their intersection could pass through the reference sphere, leading to two points straddling the occipital pole. To eliminate this possibility, one needs to select pairs of direction circles that are sure to intersect once and only once near the occipital pole, even with small perturbations of the estimated direction circles. Circles meeting at right angles or nearly so have this characteristic, and one can select gaze directions accordingly. For example, one should select pairs AB and CD in Fig. 2 rather than pairs AC and BD.

For purposes of symmetry and generality, I have described the use of five fixations, instead of the theoretical minimum of three. If one wishes to use only three fixations, two of the fixations could be collapsed onto the reference direction. A and C could be deleted, for example, and the direction circles could be calculated simply from the reference direction paired with each of the other two fixations.

EXPERIMENT

To show the validity of the technique and its internal self-consistency, a typical example of an analysis of the results will be given.

Afterimage alignments can be used to

measure the torsion of the eye in any desired position of gaze. If an eye is in the primary position and is presented with a vertical flash, the orientation of the afterimage formed on a plane will change as one shifts fixation away from the primary position, the afterimage remaining tangent, however, to a family of hyperbolic curves (Fig. 3). These curves are the projection of vertical direction circles centrally projected onto a flat screen.[8] If, however, the center of the coordination system is not coincident with the primary direction, then the orientation of afterimages will not be tangent to the family of hyperbolic curves.

For the results in Fig. 3A, we have turned the subject's bite bar and hence her head by an arbitrary amount to the right, fixing it in place. This leaves the reference direction at the center of the screen. Note that in this case the afterimage alignments are very far from being tangent to the series of hyperbolic curves: over half the settings have a discrepancy of more than 3 deg. Each afterimage setting can be expressed as a set of three Euler angles, which in turn can define a 3 x 3 matrix. See Appendix of Nakayama and Balliet.[11] From an analysis of the two pairs of oblique gaze directions or a separate analysis of two pairs of vertical and horizontal fixations, one can calculate an angular coordinate for the primary direction of regard as described above. The two open circles to the right of the origin represent two separate estimates of this calculated primary direction on the screen, one corresponding to two pairs of oblique fixations, the other corresponding to two pairs of secondary fixations. They are approximately 15.3 deg to the right of the origin of our coordinate system and are remarkably close together. Rotating the observer 15.3 deg to the left, therefore, should put the primary direction in line with the coordinate origin, and so as a check on the validity of the method, a second experiment was conducted in which the bite bar was rotated 15.3 deg to the left. Fig. 3B shows the new data obtained. Note the expected congruence of the afterimage settings to the

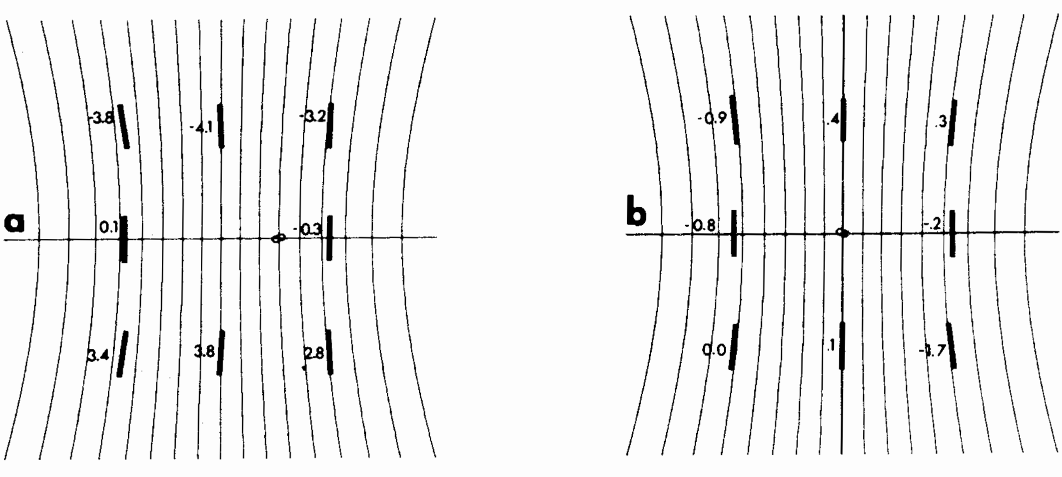

Fig. 3. Alignments of an afterimage on a planar field of fixation. If Listing's law is correct and the observer's primary position is coincident with the center of the screen, the afterimage alignments (dark bars) should be tangent to the family of hyperbolic arcs. The intersections of the hyperbolic arcs with the horizontal axis are separated by 5-deg intervals. The horizontal line of the bottom represents the eye-screen distance. Numbers next to the dark bars represent the mean deviation from Listing's law (in degrees). Each bar is the average of five settings having an average standard error of 0.25 deg. Using the center of the screen as the reference position, the calculation of the primary position was made, and the locations of two such determinations are designated by the two open circles. In a, the primary direction is not at the center of the coordinate system and, correspondingly, the afterimage alignments are not tangent to the predicted curves. After the primary direction of regard was found to be displaced 15.3 deg to the right (see open circles in a), the subject was turned 15.3 deg to the left (see b). The experiment was repeated with results showing a much closer agreement to Listing's law. Furthermore, calculations of the primary position (open circles) show it to be essentially coincident with the reference position.

family of hyperbolic arcs, showing a close agreement with Listing's law. Note further that the calculated location of the primary position under this condition is exactly where it should be: the open circles straddle the origin of the coordinate system.

DISCUSSION

Previous investigators have used a trial-and-error technique to determine the primary position. Helmholtz,[8] for example, using afterimages, searched for that position from which one could find no tilting of a vertical afterimage when looking exactly up or down and left or right, the so-called secondary positions of gaze. If the afterimage was tilted, another position was selected and the whole process was repeated until a position could be found from which there would be no tilt in secondary gaze directions—clearly a trial-and-error procedure and a lengthy one. The present method has the advantage of being noniterative and quick, requiring only that one obtain one set of measurements of the torsion in different directions of gaze. This method is especially well suited to photography, in which the film-developing time prohibits an iterative procedure.[10] Furthermore, it might also be suited for use with experimental animals when one does not wish to undertake the task of training the animals to make steady fixations to specified directions of gaze.

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